Eur. Phys. J. B **16**, 43–48 (2000)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

The use of energy and co-energy for the evaluation of forces in non-linear, anisotropic dielectric solids

S. Bobbio†

Dipartimento di Ingegneria Elettrica, via Claudio 21, Università Federico II, 80125 Napoli, Italy

Received 13 December 1999

Abstract. In this paper, in spite of the wide diffusion of the method that makes use of suitable variations of the electric energy (or co-energy) for computing forces in non-linear, anisotropic dielectric solids, it is shown that the arguments generally developed to justify this approach are, in reality, unsound. A rigorous proof of the correctness of the method is provided. It is shown also that the method is frequently used in cases where it fails; to this end, the limitations of the method itself are pointed out.

PACS. 41.20.Cv Electrostatics; Piezoelectricity and electrostriction

1 Introduction

One of the most popular methods for computing the resultant force acting upon a solid dielectric (or conducting) body immersed in a given electrostatic field makes use of the variations of the so-called electric energy (or, alternatively, of the co-energy) determined by suitable elementary, rigid displacements of the considered body.

In this paper, after having recalled briefly, in Section 2, the above mentioned method, the concept of electric energy is critically examined in Section 3, in order to show that the arguments by means of which the method itself is usually justified are, in reality, untenable. In spite of this, the approach is valid, and this is shown in Section 4, where a rigorous proof of the correctness of the method is presented.

Finally, the limitations of this approach are discussed, and the cases where it fails are pointed out.

2 The physical system

The system we shall take into consideration is general enough to allow us to handle the most common electrostatic devices (capacitors, insulators, ...). It is composed of: n conducting bodies C_1, C_2, ..., C_n, carrying free charges q_1, q_2, ..., q_n, respectively, and m solid dielectric bodies D_1, D_2, ..., D_m, devoid of free charges, immersed in free space (or, equivalently, in air).

Usually, the electric constitutive relationships of the m dielectric bodies are expressed as

$$\mathbf{D} = \mathbf{D}_j(\mathbf{E}), \quad \text{for} \quad j = 1, 2, ..., m, \tag{1}$$

$\mathbf{D}$ being the electric displacement, and $\mathbf{E}$ the electrostatic field. These relationships generally are assumed generally to be non-linear, and anisotropic.

Of course, in lieu of equations (1), the inverse relationships

$$\mathbf{E} = \mathbf{E}_j(\mathbf{D}), \quad \text{for} \quad j = 1, 2, ..., m \tag{2}$$

might be assigned equally.

As was previously anticipated, the resultant force acting upon any one of the conducting (or dielectric) bodies, say C_k, can be computed by means of suitable variations of the electric energy, or alternatively of the co-energy.

In this section, we shall recall briefly this method (in its two versions) in the form in which it is usually presented in most text books; (see, *e.g.* [1,2]).

In the next section, we shall re-examine the whole matter in detail, to discuss the soundness of the arguments generally developed to justify the method.

2.1 The method of the electric energy

In the first place, the so-called *electric energy* U_e is introduced as

$$U_\mathrm{e} = \iiint_{\Omega_\infty} \left[\int_0^{\mathbf{D}} \mathbf{E}(\mathbf{D}) \cdot \mathrm{d}\mathbf{D} \right] \mathrm{d}\Omega$$

$$= \iiint_V \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega + \sum_{j=1}^{m} \iiint_{\Omega_j} \left[\int_0^{\mathbf{D}_j} \mathbf{E}_j(\mathbf{D}) \cdot \mathrm{d}\mathbf{D} \right] \mathrm{d}\Omega_j \tag{3}$$

where: Ω_∞ is the whole space, including the regions Ω_1, Ω_2, ..., Ω_m occupied by the m dielectric bodies, V is the free space region (extending to infinity), and the integration of the electrostatic field $\mathbf{E}$ (in the variable $\mathbf{D}$) takes into account the constitutive function $\mathbf{E}_j(\mathbf{D})$ of each dielectric body. (No energy term is present for the conducting bodies, since the electrostatic field vanishes, of course, in those bodies.)

Next, an elementary, *rigid* displacement $\mathrm{d}\mathbf{x}$ of the body C_k is ideally performed, while the n *free charges* q_1, q_2, ..., q_n located on the n *conducting bodies* are *kept constant*.

The variation $\mathrm{d}U_\mathrm{e}$ is then computed in conformity with equation (3), and, finally, the component of the resultant force $\mathbf{F}_\mathrm{k}$ (exerted by the electric field upon the body C_k) in the direction of $\mathrm{d}x$ is evaluated by means of the following relationship:

$$\mathbf{F}_\mathrm{k} \cdot \mathrm{d}\mathbf{x} = -\mathrm{d}U_\mathrm{e}. \tag{4}$$

2.2 The method of the electric co-energy

The *electric co-energy* U_e^* is introduced as

$$U_\mathrm{e}^* = \iiint\limits_{\Omega_\infty} \left[\int_0^{\mathbf{E}} \mathbf{D}(\mathbf{E}) \cdot \mathrm{d}\mathbf{E}\right] \mathrm{d}\Omega$$

$$= \iiint\limits_{V} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega + \sum_{j=1}^{m} \iiint\limits_{\Omega_j} \left[\int_0^{\mathbf{E}_j} \mathbf{D}_j(\mathbf{E}) \cdot \mathrm{d}\mathbf{E}\right] \mathrm{d}\Omega_j \tag{5}$$

the meaning of the symbols being now obvious.

Next, an elementary, rigid displacement $\mathrm{d}\mathbf{x}$ of the body C_k is considered, while the potentials ϕ_1, ϕ_2, ..., ϕ_n of the n conducting bodies are kept constant (by means of suitable voltage sources).

Equation (4) is replaced, finally, by:

$$\mathbf{F}_\mathrm{k} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}U_\mathrm{e}^*. \tag{6}$$

3 Critical analysis of the method

Equation (4) is usually justified by invoking conservation of energy. In a few words, one argues roughly as follows.

To give the body C_k the rigid displacement $\mathrm{d}\mathbf{x}$, "we" should spend the elementary mechanical work

$$\delta L_\mathrm{k} = -\mathbf{F}_\mathrm{k} \cdot \mathrm{d}\mathbf{x}. \tag{7}$$

The displacement having been performed at constant free charges, no other work is spent upon the system, and then, invoking conservation of energy, one has:

$$\delta L_\mathrm{k} = \mathrm{d}U_\mathrm{e}. \tag{8}$$

It follows, therefore:

$$-\mathbf{F}_\mathrm{k} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}U_\mathrm{e},$$

as previously anticipated.

In a similar way, equation (6) is usually justified by taking into account also the work performed by the voltage sources required to keep constant the potentials ϕ_1, ϕ_2, ..., ϕ_n of the n conducting bodies during the displacement $\mathrm{d}\mathbf{x}$.

Both arguments, however, do not consider the following phenomena occurring during the displacement of C_k:

(i) the mechanical deformations induced in all solid bodies (conductors and dielectrics), by *electrostriction*;
(ii) the exchanges of *heat* between each body and its environment;
(iii) the change of *temperature* as well as of *thermodynamic internal energy* of each body.

At first sight, all these phenomena might seem negligible; in the following, however, it will be shown that neglecting them completely, as is usually claimed, leads to heavy mistakes.

Indeed, if one takes into account correctly the above mentioned phenomena, equation (8) should be replaced by the following relationship, expressing the *fundamental laws of classical thermodynamics* (for a displacement $\mathrm{d}x$ taking place at *constant free charges*):

$$\delta L_\mathrm{k} = \mathrm{d}U_f + \sum_{i=1}^{n} \mathrm{d}U_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}U_j^{(\mathrm{D})} - \sum_{i=1}^{n} T_i^{(\mathrm{C})} \mathrm{d}S_i^{(\mathrm{C})} - \sum_{j=1}^{m} T_j^{(\mathrm{D})} \mathrm{d}S_j^{(\mathrm{D})}, \tag{9}$$

where: U_f is the energy stored in the electrostatic field, $U_i^{(\mathrm{C})}$ and $U_j^{(\mathrm{D})}$ are the thermodynamic internal energies stored in the ith conducting body and in the jth dielectric body, respectively, $T_i^{(\mathrm{C})}$ and $T_j^{(\mathrm{D})}$ are the temperatures of the same bodies, and finally $S_i^{(\mathrm{C})}$ and $S_j^{(\mathrm{D})}$ are their entropies.

As previously anticipated, if one drastically simplifies the whole matter by merely neglecting all thermodynamic terms in equation (9), this relationship would be reduced to:

$$\delta L_\mathrm{k} = \mathrm{d}U_\mathrm{f}. \tag{10}$$

It should be noticed, however, that $\mathrm{d}U_\mathrm{f}$ must not be confused with $\mathrm{d}U_\mathrm{e}$ and that, therefore, equation (10) is different from equation (8).

A discussion about U_f and U_e is now in order.

By definition, the energy U_f stored in the field of a given electrostatic system corresponds to the work that "we" have to spend on "building" the system itself, starting from a situation in which all charges are diluted at infinity with an infinitesimal volume (or surface) density, and acting against the only electrostatic "long-range" forces (see, *e.g.* [3,4]). In addition, it is well known that

U_f is expressed as

$$U_\mathrm{f} = \iiint\limits_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega, \tag{11}$$

where ε_0 is the permittivity of free space, and the integration is extended over the whole space, including the regions occupied by material bodies.

Turning now to the so-called electric energy U_e, some fundamental points should be stressed firstly:

(i) the electric constitutive relationship of each solid dielectric body is not so simple as equations (2) or (3), but is generally of the following kind:

$$\mathbf{D} = \mathbf{D}_j(\mathbb{E}, \mathbf{E}, T), \quad \text{for} \quad j = 1, 2, ..., m, \tag{12}$$

$\mathbb{E}$ being the strain tensor (expressing the state of mechanical deformation of the body), and T the (absolute) temperature of the body itself. Of course, the constitutive relationships might be expressed, equivalently, as

$$\mathbf{E} = \mathbf{E}_j(\mathbb{E}, \mathbf{D}, T), \quad \text{for} \quad j = 1, 2, ..., m. \tag{13}$$

It follows that, when the integration of $\mathbf{E}$ (in the variable $\mathbf{D}$), appearing in equation (3), and the integration of $\mathbf{D}$ (in the variable $\mathbf{E}$), appearing in equation (5), are performed, the variations of the strain tensor $\mathbb{E}$ as well as of the temperature T along the integration path should be simultaneously specified; otherwise, the integrals would remain undetermined;

(ii) once the variations of $\mathbb{E}$ and T along the integration paths have been specified, U_e, as expressed by equation (3), represents the total work that "we" have to spend in the course of the considered process, starting from $\mathbf{D} = \mathbf{0}$ in each point P of the whole space and ending to $\mathbf{D}(P)$; (see, *e.g.* [3]);

(iii) we cannot be sure *a priori*, however, that this work is entirely transformed in energy stored in the electrostatic field, since part of it could be converted in a) variations of the internal energies of material bodies, b) heat transferred to the environment.

Now, expressing $\mathbf{D}$, in dielectric bodies, as

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P},$$

($\mathbf{P}$ being the polarization density), and taking into account the "true" electric constitutive relationships of the dielectric bodies, equations (12, 13), one has:

$$U_\mathrm{e} = \iiint\limits_{V} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega + \sum_{j=1}^{m} \iiint\limits_{\Omega_i} \left[\int_0^{\mathbf{D}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{D}, T) \cdot \mathrm{d}(\varepsilon_0 \mathbf{E} + \mathbf{P}) \right] \mathrm{d}\Omega_j,$$

and then:

$$U_\mathrm{e} = \iiint\limits_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega + \sum_{j=1}^{m} \iiint\limits_{\Omega_i} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j, \tag{14}$$

having now indicated as $\mathbf{E}_j(\mathbb{E}, \mathbf{P}, T)$ the relationship obtained when $\mathbf{D}$ is replaced by $\varepsilon_0 \mathbf{E} + \mathbf{P}$ into equation (13).

One has, therefore:

$$U_\mathrm{e} - U_\mathrm{f} = \sum_{j=1}^{m} \iiint\limits_{\Omega_i} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j, \tag{15}$$

and must conclude that U_e does not represent the energy stored in the electrostatic field, but a mixed term containing this energy as well as another term concerning the specific polarization process taken into consideration (see, *e.g.* [4,5]). This shows that equation (4) does not correspond to merely neglecting all thermodynamic terms in equation (9), since this would lead to equation (10), which is different from equation (4).

In particular, it is known that, when the integration of $\mathbf{E}_j$ in the variable $\mathbf{P}$ is performed at constant deformation and temperature (*i.e.*, at constant $\mathbb{E}$ and T), each term of the sum appearing on the r.h.s. of equation (15) represents the variation of the Helmholtz free energy in one of the dielectric bodies, in the course of a polarization process (starting from $\mathbf{P} = \mathbf{0}$ and ending to $\mathbf{P}_j$) taking place at constant deformation and temperature (see [3,4]).

The previous considerations show that the the arguments generally used to justify the method of computing forces through the variations of the so-called electric energy are, in reality, unsound. Similar conclusions hold also as far as the method of co-energy is concerned.

In spite of this, both methods are correct, as will be shown in the next section.

4 A rigorous justification of the two methods

In this section we shall provide a rigorous proof of the correctness of the two methods. At the same time, we shall point out their main limitations.

4.1 The method of the electric energy

Recalling equation (9), and assuming that all material bodies have initially the same temperature T, one has:

$$\delta L_\mathrm{k} = \mathrm{d}U_\mathrm{f} + \sum_{i=1}^{n} \mathrm{d}U_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}U_j^{(\mathrm{D})} - \sum_{i=1}^{n} T\,\mathrm{d}S_i^{(\mathrm{C})} - \sum_{j=1}^{m} T\,\mathrm{d}S_j^{(\mathrm{D})}. \tag{16}$$

Let us introduce now the Helmholtz free energies of all conducting and dielectric bodies:

$$\begin{aligned} A_1^{(\mathrm{C})} &= U_1^{(\mathrm{C})} - TS_1^{(\mathrm{C})}, \\ &\cdots\cdots\cdots\cdots \\ A_n^{(\mathrm{C})} &= U_n^{(\mathrm{C})} - TS_n^{(\mathrm{C})}, \\ A_1^{(\mathrm{D})} &= U_1^{(\mathrm{D})} - TS_1^{(\mathrm{D})}, \\ &\cdots\cdots\cdots\cdots \\ A_m^{(\mathrm{D})} &= U_m^{(\mathrm{D})} - TS_m^{(\mathrm{D})}. \end{aligned} \tag{17}$$

Equation (16) can be rewritten, then, as:

$$\begin{aligned} \delta L_{\mathrm{k}} = \mathrm{d}U_{\mathrm{f}} &+ \sum_{i=1}^{n} \mathrm{d}A_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\mathrm{D})} \\ &+ \sum_{i=1}^{n} S_i^{(\mathrm{C})} \mathrm{d}T_i^{(\mathrm{C})} + \sum_{j=1}^{m} S_j^{(\mathrm{D})} \mathrm{d}T_j^{(\mathrm{D})}. \end{aligned} \tag{18}$$

On the other hand, for a given initial configuration of the bodies, and for assigned free charges q_1, q_2, ..., q_n, the resultant force $\mathbf{F}_{\mathrm{k}}$ is uniquely determined. This implies, in addition, that also the elementary work δL_{k} is uniquely determined for a prescribed displacement d$\mathbf{x}$.

Thus, if we assume that the temperature of all bodies is held constant during the displacement of C_{k}, (which is in any case possible having recourse to a suitable heat reservoir), equation (18) becomes:

$$\delta L_{\mathrm{k}} = \mathrm{d}U_{\mathrm{f}} + \sum_{i=1}^{n} \mathrm{d}A_i^{(\mathrm{C})} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\mathrm{D})}, \tag{19}$$

having now indicated as $\mathrm{d}U_{\mathrm{f}}$, $\mathrm{d}A_i^{(\mathrm{C})}$ and $\mathrm{d}A_j^{(\mathrm{D})}$ the variations of the energy stored in the electric field and the changes of the Helmholtz free energies of the bodies, induced by a displacement d$\mathbf{x}$ taking place at constant free charges and temperature.

Let us notice now that, δL_{k} being uniquely determined, the r.h.s. of equation (19) is also assigned for a given initial configuration of the system, and for a prescribed displacement d$\mathbf{x}$.

On the other hand, one can rightfully imagine of replacing the n conducting bodies with others having different elastic properties, and the m solid dielectric bodies with other bodies having the same electric constitutive relationships but different elastic characteristics.

In such a case, each body would be subjected to a different deformation with respect to the previous situation, and each term in the sum appearing on the r.h.s. of equation (19) would be different in comparison with the previous corresponding term, whereas the total sum should remain unchanged.

This circumstance offers us the opportunity of computing δL_{k} in a very simple way, by merely assuming that all bodies (conductors as well as dielectrics) are mechanically rigid and remain then undeformed during the displacement d$\mathbf{x}$. In such a case, one has:

$$\mathrm{d}A_i^{(\mathrm{C})} = 0, \quad \text{for} \quad i = 1, 2, ..., n,$$

since no variation of the Helmholtz free energy can arise in a conducting body remaining undeformed at constant temperature.

In addition, recalling that the term

$$\iiint_{\Omega_i} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j$$

corresponds precisely to the variation of the Helmholtz free energy of the jth dielectric body in the course of a process of polarization taking place at constant deformation and temperature, one concludes easily that

$$\mathrm{d}A_j^{(\mathrm{D})} = \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j,$$

and then:

$$\begin{aligned} \delta L_{\mathrm{k}} = \mathrm{d}U_{\mathrm{f}} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\mathrm{D})} &= \mathrm{d} \left[\iiint_{\Omega_\infty} \frac{1}{2} \varepsilon_0 E^2 \mathrm{d}\Omega \right] \\ &+ \sum_{j=1}^{m} \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j. \end{aligned} \tag{20}$$

Finally, recalling equation (14), one has:

$$\delta L_{\mathrm{k}} = \mathrm{d}U_{\mathrm{e}},$$

as previously anticipated.

Equation (20) proves that the electric energy method of computing forces is rigorously correct, provided that the rigid, elementary displacement d$\mathbf{x}$ is subjected to the following constraints:

(i) the free charges located on the conducting bodies are kept constant;
(ii) the temperature of all bodies is held constant;
(iii) the state of deformation of each body remains unchanged, (*i.e.*, all bodies are considered as rigid).

Notice that the above mentioned constraints are precisely the conditions respected by the naïve application of the method.

4.2 The method of the electric co-energy

The main difference between the method of co-energy with respect to the approach making use of energy is that the potentials of the n conducting bodies are kept constant, instead of free charges, during the elementary displacement d$\mathbf{x}$.

For this purpose, suitable voltage sources must be ideally connected to the n conducting bodies. On the other

hand, these sources will perform some elementary work $\delta L_{\rm vs}$ during the displacement $\mathrm{d}\mathbf{x}$ (momentarily, we do not need the expression of $\delta L_{\rm vs}$: we shall provide it later, at the right moment).

However, equation (9), expressing the fundamental laws of thermodynamics, should be modified now, to account for the new work term. We must write, therefore:

$$\delta L_{\rm k} + \delta L_{\rm vs} = \mathrm{d}U_{\rm f} + \sum_{i=1}^{n} \mathrm{d}U_i^{(\rm C)} + \sum_{j=1}^{m} \mathrm{d}U_j^{(\rm D)} - \sum_{i=1}^{n} T_i^{(\rm C)} \mathrm{d}S_i^{(\rm C)} + \sum_{j=1}^{m} T_j^{(\rm D)} \mathrm{d}S_j^{(\rm D)}. \quad (21)$$

Assuming, in addition, that all bodies have initially the same temperature T, and introducing the Helmholtz free energies given by equations (17), (21) becomes:

$$\delta L_{\rm k} + \delta L_{\rm vs} = \mathrm{d}U_{\rm f} + \sum_{i=1}^{n} \mathrm{d}A_i^{(\rm C)} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\rm D)} + \sum_{i=1}^{n} S_i^{(\rm C)} \mathrm{d}T_i^{(\rm C)} + \sum_{j=1}^{m} S_j^{(\rm D)} \mathrm{d}T_j^{(\rm D)}. \quad (22)$$

Arguments similar in all respects to those developed in Section 4.1 lead now to

$$\delta L_{\rm k} + \delta L_{\rm vs} = \mathrm{d}U_{\rm f} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\rm D)} = \mathrm{d}\left[\iiint_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega \right] + \sum_{j=1}^{m} \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j. \quad (23)$$

The time has come now of expressing $\delta L_{\rm vs}$. One is easily convinced that

$$\delta L_{\rm vs} = \sum_{i=1}^{n} \phi_i \mathrm{d}q_i,$$

having indicated as $\mathrm{d}q_i$, for $i = 1, 2, ..., n$, the variations of the free charges of the n conducting bodies required to keep constant their potentials.

On the other hand, one has easily (see, *e.g.* [3]):

$$\iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{D} \, \mathrm{d}\Omega = \sum_{i=1}^{n} \phi_i q_i.$$

Thus, one concludes that the elementary work $\delta L_{\rm vs}$ performed by the voltage sources can be expressed as

$$\delta L_{\rm vs} = \sum_{i=1}^{n} \phi_i \mathrm{d}q_i = \mathrm{d} \sum_{i=1}^{n} \phi_i q_i = \mathrm{d} \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{D} \, \mathrm{d}\Omega, \quad (24)$$

provided that the term

$$\mathrm{d} \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{D} \mathrm{d}\Omega$$

is evaluated keeping constant the potentials of the n conducting bodies.

Now, replacing the expression of $\delta L_{\rm vs}$ given by equation (24) into equation (23), one has:

$$\delta L_{\rm k} + \mathrm{d} \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{D} \mathrm{d}\Omega = \mathrm{d}U_{\rm f} + \sum_{j=1}^{m} \mathrm{d}A_j^{(\rm D)}$$

$$= \mathrm{d}\left[\iiint_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega \right] + \sum_{j=1}^{m} \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j,$$

and then:

$$\delta L_{\rm k} = \mathrm{d}\left[\iiint_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega \right] + \sum_{j=1}^{m} \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j - \mathrm{d} \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{D} \mathrm{d}\Omega$$

$$= -\mathrm{d}\left[\iiint_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega \right] + \sum_{j=1}^{m} \mathrm{d} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j - \mathrm{d} \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{P} \, \mathrm{d}\Omega$$

$$= -\mathrm{d}\left[\iiint_{\Omega_\infty} \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}\Omega \right] - \mathrm{d}\left\{ \iiint_{\Omega_\infty} \mathbf{E} \cdot \mathbf{P} \, \mathrm{d}\Omega - \sum_{j=1}^{m} \iiint_{\Omega_j} \left[\int_0^{\mathbf{P}_j} \mathbf{E}_j(\mathbb{E}, \mathbf{P}, T) \cdot \mathrm{d}\mathbf{P} \right] \mathrm{d}\Omega_j \right\}$$

$$= -\iiint_{\Omega_\infty} \left[\int_0^{\mathbf{E}} \mathbf{D}(\mathbb{E}, \mathbf{E}, \mathbf{T}) \cdot \mathrm{d}\mathbf{E} \right] \mathrm{d}\Omega = -\mathrm{d}U_{\rm e}^*. \quad (25)$$

We conclude, therefore, that

$$\mathbf{F}_{\mathrm{k}} \cdot \mathrm{d}\mathbf{x} = \iiint_{\Omega_\infty} \left[\int_0^{\mathbf{E}} \mathbf{D}(\mathbb{E},\, \mathbf{E},\, \mathbf{T}) \cdot \mathrm{d}\mathbf{E} \right] \mathrm{d}\Omega = \mathrm{d}U_{\mathrm{e}}^{*}, \tag{26}$$

provided that the integration of $\mathbf{D}$ in the variable $\mathbf{E}$ is performed for constant potentials of the n conducting bodies, and keeping constant the temperature as well as the state of deformation of all bodies (*i.e.*, considering them as rigid).

Before ending this section, it is worth to point out that the above discussed methods are frequently utilized even when, in reality, they fail.

Indeed, the two methods of computing forces require, to be correct, two conditions:
(i) the solid dielectric bodies must be non-hysteretic; otherwise, the term appearing on the r.h.s. of equation (15) would not represent the variation of the Helmholtz free energy stored in the dielectric bodies, (since a fraction of this term would be dissipated in heat and would not contribute to the mechanical work);
(ii) the electric constitutive relationships of the solid dielectrics, expressed by the equations (12), should be such that the integrals of $\mathbf{E}$ in the variable $\mathbf{D}$ are independent of the way in which $\mathbf{D}$ goes from zero to its final value (at constant temperature and deformation). This requires that the Jacobian matrix of the functions $\mathbf{D}_j(\mathbb{E}, \mathbf{E}, T)$ with respect to the variable $\mathbf{E}$ (for constant $\mathbb{E}$ and T) is symmetric. Typical is the case of electrically linear dielectrics, which require that the tensorial dielectric constant is symmetric.

5 Conclusions

The concept of electric energy in non-linear, anisotropic dielectric solids has been critically analysed and has been shown to be expressed by the sum of two terms:
(i) the energy stored in the electrostatic field,
(ii) the variation of the Helmholtz free energy of the dielectric body, when subjected to a polarization process taking place at constant temperature and deformation.

On this ground, the method of computing forces through the variations of the electric energy (or co-energy) has been critically analysed and it has been shown that the arguments usually developed for justifying such a method are, in reality, unsound.

A rigorous proof of the correctness of the method is then presented. Finally, the limitations of this approach have been pointed out.

References

1. H.A. Haus, J.R. Melcher, *Electromagnetic fields and energy* (Prentice-Hall, Englewood Cliffs, N.J., 1989).
2. J.D. Jackson, *Classical electrodynamics* (John Wiley & Sons, New York, 1962).
3. L.D. Landau, E.M. Lifshitz, *Electrodynamique des milieux continus* (translated from the russian edition appeared in 1959, MIR, Moscow).
4. S. Bobbio, *Electrodynamics of materials; forces, stresses and energies in solids and fluids* (Academic Press, New York, 1999).
5. R.M. Fano, L.J. Chu, R.B. Adler, *Electromagnetic fields, energy, and forces* (John Wiley & Sons, New York, 1959).